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Remark on the Branching Theorem and Supersymmetric Algebras

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INTRODUCTION

Supersymmetric algebras have already proved useful in giving transparent proofs of a number of basic results of representation theory [3, 7–9, 11–15]. Specifically, the technique of introducing virtual variables, which may have a different signature than the signature of variables to be delt with, often cuts down the amount of computation. Furthermore, the extension of results of representation theory to the superalgebraic setting sheds new light, and permits us to establish natural correspondences that were formally missing.

In this note we carry out this program by deriving a superalgebraic version of the Branching Rules for the representations of the general linear group. While the statement of supersymmetric branching rules is in all respects similar to the ordinary one (and differing from it in our allowing variables of two signature), the proof yields a useful dividend, namely, a simple combinatorial construction of a canonical basis for the decomposition of a restriction of a representation.

As an application we give a supersymmetric generalization of Pieri's formula, as well as a proof of this formula which is perhaps as short as it can be whittled down to. This application has been inspired by some recent work of Boffi [5, 6].

We have benefited from the pioneering work of Berele and Regev [3], who were first to state such a supersymmetric extension of branching rules, as well as from the insights of Balentekin and Bars [2].

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In order to make the exposition self contained, in Sections 1 and 2 we summarize some basic facts from [13, 3, 7, 8]. The results we establish are explictly formulated as twofold statements, even though, with regard to decomposition results, each of the two formulations could be derived from the other by a natural isomorphism (Section 6).

1. Superalgebras and Polarization Operators

A *virtual $\mathbb{Z}_2$-graded set* is a pair $(E, |\ |)$, where E is a set and $|\ |: X \to \mathbb{Z}_2$ is a map such that the fibers $E_i = \{x \in E;\ |x| = i\}$, $i \in \mathbb{Z}_2$, are countable sets. In the following, we will write $|n|$ for the parity of the integer n, that is, its class modulo 2. Let $\mathrm{Mon}(E)$ be the free monoid generated by the set E; the *parity* of a word $\omega \in \mathrm{Mon}(E)$, $\omega = x_1 x_2 \cdots x_n$, $x_i \in E$, is defined by setting

$$|\omega| = \sum_{i=1}^{n} |x_i| \in \mathbb{Z}_2.$$

Given an element $e \in E$ and a word $\omega \in \mathrm{Mon}(E)$, the *content of ω with respect to e* is the number $\mathrm{cont}(\omega; e)$ of entries of ω which equal e.

Let $\mathbb{K}$ be an arbitrary field of characteristic zero that will remain fixed throughout this paper. The supersymmetric algebra $\mathrm{Super}[E]$ is the free $\mathbb{Z}_2$-graded commutative $\mathbb{K}$-algebra generated by E, that is the quotient algebra of the semigroup $\mathbb{K}$-algebra of $\mathrm{Mon}(E)$ with respect to the ideal J generated by the elements

$$uv - (-1)^{|u| \cdot |v|}\, vu, \qquad u, v \in \mathrm{Mon}(E).$$

$\mathrm{Super}[E]$ is an associative superalgebra with respect to the decomposition

$$\mathrm{Super}[E] = \mathrm{Super}[E]_0 \oplus \mathrm{Super}[E]_1,$$

where $\mathrm{Super}[E]_i$ is the $\mathbb{K}$-linear span of all words $\omega \in \mathrm{Mon}(E)$ such that $|\omega| = i$, $i \in \mathbb{Z}_2$.

For $\alpha \in \mathbb{Z}_2$, an element $L \in \mathrm{End}_{\mathbb{K}}(\mathrm{Super}[E])$ is said to be an *α-homogeneous left superderivation* if the following conditions hold:

(i) $L[\mathrm{Super}[E]_i] \subseteq \mathrm{Super}[E]_{i+\alpha}$, $i \in \mathbb{Z}_2$

(ii) $L(uv) = L(u)\, v + (-1)^{\alpha \cdot |u|}\, uL(v)$, $u, v \in \mathrm{Mon}(E)$.

Similarly, if $R \in \mathrm{End}_{\mathbb{K}}(\mathrm{Super}[E])$ satisfies (i) and

(ii′) $R(uv) = (-1)^{\alpha \cdot |v|}\, R(u)v + uR(v)$, $u, v \in \mathrm{Mon}(E)$,

R is said to be an *α-homogeneous right superderivation.*

A left (right) superderivation is a linear combination of homogeneous left (right) superderivations. From now on, a right superderivation will be written on the right of its argument; products of right superderivations will be considered left associative; that is, given an element $\omega \in \mathrm{Super}[E]$, we write $(\omega) R_1 R_2$ for $((\omega) R_1) R_2$.

The following commutation property will play a crucial role in the sequel.

PROPOSITION 1. *Let L and R be a left and a right superderivation of* $\mathrm{Super}[E]$, *respectively. Then we have*

$$(L(\omega)) R = L((\omega) R)$$

for every $\omega \in \mathrm{Super}[E]$.

Given two virtual $\mathbb{Z}_2$-graded sets X, whose elements are called *letters*, and Y, whose elements are called *places*, we define the *letterplace* set $X \times Y$ to be the set $\{(x \mid y); x \in X, y \in Y\}$. Setting $|(x \mid y)| = |x| + |y|$ the letterplace set $X \times Y$ becomes a virtual $\mathbb{Z}_2$-graded set. We write $\mathrm{Super}[X \mid Y]$ for the supersymmetric algebra generated by $X \times Y$; this superalgebra is called the (virtual) *letterplace superalgebra.*

Given two letters x_i, x_j, the *letter polarization operator* $D_{x_i x_j}$ is the unique α-homogeneous left superderivation of $\mathrm{Super}[X \mid Y]$, with $\alpha = |x_i| + |x_j|$, such that

$$D_{x_i x_j}(z \mid y) = \delta_{x_j z}(x_i \mid y)$$

for every $z \in X$, $y \in Y$.

Given two places y_h, y_k, *the place polarization operator* ${}_{y_h y_k}\mathfrak{D}$ is the unique α-homogeneous right superderivation of $\mathrm{Super}[X \mid Y]$, with $\alpha = |y_h| + |y_k|$, such that

$$(x \mid t)\, {}_{y_h y_k}\mathfrak{D} = \delta_{t y_h}(x \mid y_k)$$

for every $x \in X$, $t \in Y$.

2. REPRESENTATION THEORY

A Young diagram $T = (\omega_1, \omega_2, ..., \omega_p)$ on the set E is a sequence of words $\omega_i \in \mathrm{Mon}(E)$ such that $\mathrm{length}(\omega_1) \geqslant \mathrm{length}(\omega_2) \geqslant \cdots \geqslant \mathrm{length}(\omega_p) > 0$. The *shape* of T, $\mathrm{sh}(T)$, is the vector $\lambda = (\lambda_1, ..., \lambda_p)$, where $\lambda_i = \mathrm{length}(\omega_i)$; we set, by convention, $\lambda_i = 0$ for every $i > p$ and call the integer $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_p$ the *total content* of the diagram T. The *content of T with respect to $e \in E$* is the integer $\mathrm{cont}(T; e) = \sum_{i=1}^{p} \mathrm{cont}(\omega_i; e)$.

If $\omega_i = x_{i1}x_{i2}\cdots x_{i\lambda_i}$, $x_{ij}\in E$, the words $\tilde{\omega}_j = x_{1j}x_{2j}\cdots x_{\tilde{\lambda}_j j}$, where $\tilde{\lambda}_j$ equals the number of words ω_i of T such that length$(\omega_i)\geqslant j$, define the conjugate Young diagram $\tilde{T}=(\tilde{\omega}_1, \tilde{\omega}_2, ..., \tilde{\omega}_q)$; its shape $\tilde{\lambda}=(\tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_q)$, with $q=\lambda_1$, is the conjugate shape of λ.

Let now a linear order $\leqslant$ be given on the $\mathbb{Z}_2$-graded set E; the diagram T is said to be $\leqslant$-*standard* (or simply *standard*, when no confusion could arise about linear orders) whenever the following conditions are met:

(i) if $|x_{ih}|=0$, then $x_{ih}\leqslant x_{i,h+1}$,

(ii) if $|x_{ih}|=1$, then $x_{ih}< x_{i,h+1}$,

(iii) if $|x_{kj}|=0$, then $x_{kj}< x_{k+1,j}$,

(iv) if $|x_{kj}|=1$, then $x_{kj}\leqslant x_{k+1,j}$,

for every $i=1, 2, ..., p$; $j=1, 2, ..., \lambda_1$; $h=1, 2, ..., \lambda_p-1$; $k=1, 2, ..., \tilde{\lambda}_j-1$.

Given p distinct symbols $a_1, ..., a_p\in E$, the diagram $(\omega_1, \omega_2, ..., \omega_p)$ such that $\omega_i=a_i^{\lambda_i}$ will be denoted by Coder$(\lambda; \{a_i\})$; similarly, we write Der$(\lambda; \{a_i\})$ for the conjugate diagram of Coder$(\tilde{\lambda}; \{a_i\})$.

Given two words $\omega\in \mathrm{Mon}(X)$, $\bar{\omega}\in\mathrm{Mon}(Y)$, length$(\omega)=$ length$(\bar{\omega})=n$, $\omega=x_1x_2\cdots x_n$, $\bar{\omega}=y_1y_2\cdots y_n$, we write

$$\langle\omega\,|\,\bar{\omega}\rangle$$

for the monomial

$$(x_1\,|\,y_1)(x_2\,|\,y_2)\cdots(x_n\,|\,y_n)$$

in Super$[X|\,Y]$.

Given $\alpha\in X$, we write $D_{\omega\alpha^n}$ for the product of (left) linear operators

$$D_{x_1\alpha}D_{x_2\alpha}\cdots D_{x_n\alpha};$$

similarly, given $\beta\in Y$, we write ${}_{\beta^n\bar{\omega}}\reflectbox{D}$ for the product of (right) linear operators

$${}_{\beta y_1}\reflectbox{D}\;{}_{\beta y_2}\reflectbox{D}\cdots{}_{\beta y_n}\reflectbox{D}.$$

Let $S=(\omega_1, \omega_2, ..., \omega_p)$, $T=(\bar{\omega}_1, \bar{\omega}_2, ..., \bar{\omega}_p)$ be Young diagrams on X and Y, respectively, with sh$(S)=$ sh(T). Let $\alpha_1, ..., \alpha_p\in X_0$, $\beta_1, ..., \beta_p\in Y_0$ be symbols such that

$$\mathrm{cont}(\omega_j;\alpha_i)=\mathrm{cont}(\bar{\omega}_j;\beta_i)=0,$$

for every $i, j=1, 2, ..., p$. The *bitableau* $(S\,|\,T)$ is the element

$$\begin{aligned}(S\,|\,T) &= D_{\omega_1\alpha_1^{\lambda_1}}\cdots D_{\omega_p\alpha_p^{\lambda_p}}\left(\langle\alpha_1^{\lambda_1}\,|\,\bar{\omega}_1\rangle\cdots\langle\alpha_p^{\lambda_p}\,|\,\bar{\omega}_p\rangle\right)\\ &= \left(\langle\omega_1\,|\,\beta_1^{\lambda_1}\rangle\cdots\langle\omega_p\,|\,\beta_p^{\lambda_p}\rangle\right)\,{}_{\beta_1^{\lambda_1}\bar{\omega}_1}\reflectbox{D}\cdots{}_{\beta_p^{\lambda_p}\bar{\omega}_p}\reflectbox{D}\end{aligned}$$

of Super$[X|\,Y]$. If sh$(S)\neq$ sh(T), we set $(S\,|\,T)=0$.

We explicitly remark that the choice of the symbols α_i, β_i is irrelevant. Let $\gamma_1, ..., \gamma_q \in X_1$, $q = \lambda_1$, and $\delta_1, ..., \delta_p \in Y_0$ be symbols such that

$$\operatorname{cont}(\omega_j; y_h) = \operatorname{cont}(\bar{\omega}_j; \delta_i) = 0, \qquad i, j = 1, ..., p, \qquad h = 1, ..., q.$$

The *left symmetrized bitableau* $(\boxed{S} \mid T)$ is defined by setting

$$\begin{aligned}(\boxed{S} \mid T) &= D_{\tilde{\omega}_1 \gamma_1^{\lambda_1}} \cdots D_{\tilde{\omega}_q \gamma_q^{\lambda_q}} (\operatorname{Der}(\lambda; \{\gamma_i\}) \mid T) \\ &= (\lambda_1 ! \lambda_2 ! \cdots \lambda_p !)^{-1} \\ &\qquad \times D_{\tilde{\omega}_1 \gamma_1^{\lambda_1}} \cdots D_{\tilde{\omega}_q \gamma_q^{\lambda_q}} (\operatorname{Der}(\lambda; \{\gamma_i\}) \mid \operatorname{Coder}(\lambda; \{\delta_j\}))\, {}_{\delta_1^{\lambda_1} \bar{\omega}_1}D \cdots {}_{\delta_p^{\lambda_p} \bar{\omega}_p}D .\end{aligned}$$

Let now L be a finite subset of X, p and q be the cardinalities of L_0 and L_1, respectively; the *hook set* $H(L)$ is the set of all shapes λ such that $\lambda_{p+1} \leqslant q$. We will denote by $\operatorname{Tab}_\lambda(L)$ the set of all Young diagrams $T = (\omega_1, \omega_2, ..., \omega_p)$, $\operatorname{sh}(T) = \lambda$, such that $\omega_i \in \operatorname{Mon}(L)$ for every i.

Given a shape $\lambda \in H(L)$, λ_1 symbols $\beta_1, ..., \beta_{\lambda_1} \in Y_1$, let us consider the $\mathbb{K}$-subspace of $\operatorname{Super}[X \mid Y]$ spanned by the set of all bitableaux

$$(T \mid \operatorname{Der}(\lambda; \{\beta_i\}))$$

with $T = (\omega_1, \omega_2, ..., \omega_p)$, $\omega_i \in \operatorname{Mon}(L)$; this subspace is called the *Schur module of shape λ over the set L* and it is denoted by $S_\lambda(L)$. We explicitly note that the choice of the β_i's is irrelevant, up to trivial isomorphisms; hence, we will frequently write $\operatorname{Der}_1 \lambda$ for the diagram $\operatorname{Der}(\lambda; \{\beta_i\})$, the choice of a set of elements $\beta_1, ..., \beta_{\lambda_1} \in Y_1$ being understood.

PROPOSITION 2. *Let $\lambda \in H(L)$:*

(i) $S_\lambda(L) \neq (0)$.

(ii) *$S_\lambda(L)$ is the linear span of the set of all left symmetrized bitableaux $(\boxed{Z} \mid \operatorname{Der}_1 \lambda)$, with $Z \in \operatorname{Tab}_\lambda(L)$.*

(iii) *The set $\{(S \mid \operatorname{Der}_1 \lambda); S \in \operatorname{Tab}_\lambda(L)$, S standard$\}$ is a basis of $S_\lambda(L)$.*

(iv) *The set $\{(\boxed{S} \mid \operatorname{Der}_1 \lambda); S \in \operatorname{Tab}_\lambda(L)$, S standard$\}$ is a basis of $S_\lambda(L)$.*

In the following, we will denote by $pl(L)$ the Lie superalgebra with basis

$$\{E_{x_i x_j}; x_i, x_j \in L\}$$

whose commutator is defined by extending bilinearly the map $[\ ,\ ]$: $pl(L) \times pl(L) \to pl(L)$, such that

$$[E_{x_i x_j}, E_{x_h x_k}] = \delta_{j,h} E_{x_i x_k} - (-1)^{(|x_i| + |x_j|)(|x_h| + |x_k|)} \delta_{i,k} E_{x_h x_j},$$

$x_i, x_j, x_h, x_k \in L$.

The Lie superalgebra $pl(L)$ can be identified with the *general linear Lie superalgebra* $pl(V)$ of all the endomorphisms of a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$, $\dim(V_0) = p$, $\dim(V_1) = q$. Furthermore, the map

$$E_{x_i x_j} \mapsto D_{x_i x_j}$$

uniquely defines an even map of Lie superalgebras from $pl(L)$ to the general linear Lie superalgebra of $\mathrm{Super}[X \mid Y]$ and, hence, it defines a $pl(L)$-module $pl(L) \cdot \mathrm{Super}[X \mid Y]$ on the letterplace superalgebra.

The term *Schur module* for the subspace $S_\lambda(L)$ is justified by the following result.

PROPOSITION 3. *Let* $\lambda \in H(L)$. $S_\lambda(L)$ *is a simple* $pl(L)$*-submodule of* $\mathrm{Super}[X \mid Y]$.

3. LOCALLY SYMMETRIZED BITABLEAUX

Given two shapes $\mu = (\mu_1, ..., \mu_q)$, $\lambda = (\lambda_1, ..., \lambda_p)$, $q \leqslant p$, we will write $\mu \subseteq \lambda$ whenever $\mu_i \leqslant \lambda_i$ for every $i = 1, 2, ..., q$. Let L be a proper $\mathbb{Z}_2$-graded set, μ, λ shapes such that $\mu \subseteq \lambda$, $\alpha_1, ..., \alpha_{\mu_1} \in X - L$, $|\alpha_i| = 1$ for every $i = 1, ..., \mu_1$; given a diagram $Z = (\omega_1, \omega_2, ..., \omega_q) \in \mathrm{Tab}_\mu(L)$, a diagram $Z' \in \mathrm{Tab}_\lambda(Y)$, and an element $x \in X - \{\alpha_1, ..., \alpha_{\mu_1}\}$, we define the *locally symmetrized bitableau*

$$(\boxed{Z}_{\mu x} \mid Z')$$

to be the element

$$D_{\tilde{\omega}_1 \alpha_1^{\tilde{\mu}_1}} \cdots D_{\tilde{\omega}_{\mu_1} \alpha_{\mu_1}^{\tilde{\mu}_t}} (\tilde{X} \mid Z'), \qquad t = \mu_1.$$

of $\mathrm{Super}[X \mid Y]$, where $X = (\mathbf{x}_1, ..., \mathbf{x}_{\lambda_1})$ is the diagram of shape $\tilde{\lambda}$ such that

$$\mathbf{x}_i = \alpha_i^{\tilde{\mu}_i} x^{\tilde{\lambda}_i - \tilde{\mu}_i}, \qquad \text{for every} \quad i = 1, 2, ..., \lambda_1.$$

We note again that the choice of the elements $\alpha_i \in X - L$ is irrelevant.

EXAMPLE 1. Set $\mu = (2, 2, 1)$, $\lambda = (3, 3, 2)$, $Z = (ab, bc, b) \in \mathrm{Tab}_\mu(\{a, b, c\})$. Then, we have

$$(\boxed{Z}_{\mu x} \mid \mathrm{Der}_1 \lambda) = D_{a\alpha_1} D_{b\alpha_1}^2 D_{b\alpha_2} D_{c\alpha_2} \left(\begin{array}{ccc|ccc} \alpha_1 & \alpha_2 & x & \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \alpha_2 & x & \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & x & & \beta_1 & \beta_2 & \end{array} \right).$$

In the following, L will denote a fixed $\mathbb{Z}_2$-graded proper set; given $x \in X - L$, we set $L' = L \cup \{x\}$. Furthermore, $\mu = (\mu_1, ..., \mu_q)$, $\lambda = (\lambda_1, ..., \lambda_p)$, will be shapes such that $\mu \subseteq \lambda$, $\lambda \in H(L')$.

PROPOSITION 4. $(\boxed{Z}_{\mu x} | \mathrm{Der}_1 \lambda) \in S_\lambda(L')$, *for every diagram* $Z \in \mathrm{Tab}_\mu(L)$.

We recall that the Schur module $S_\lambda(L')$ is a simple $pl(L')$-module; on the other hand, $S_\lambda(L')$ turns out to be a $pl(L)$-module, under the identity monomorphism $pl(L) \hookrightarrow pl(L')$. We have:

THEOREM 1. (i) *The map* $(\boxed{Z} | \mathrm{Der}_1 \mu) \mapsto (\boxed{Z}_{\mu x} | \mathrm{Der}_1 \lambda)$, $Z \in \mathrm{Tab}_\mu(L)$, *uniquely defines a* $pl(L)$*-module homomorphism*

$$\mathscr{S}_{\mu x}: S_\mu(L) \to S_\lambda(L').$$

(ii) *Let* $|x| = 0$. *If* $max\{\tilde{\lambda}_i - \tilde{\mu}_i;\ i = 1, 2, ..., \lambda_1\} \leqslant 1$, *then* $\mathscr{S}_{\mu x}$ *is injective.*

(iii) *Let* $|x| = 1$. *If* $\max\{\lambda_i - \mu_i;\ i = 1, 2, ..., p\} \leqslant 1$, *then* $\mathscr{S}_{\mu x}$ *is injective.*

Proof. Let $\delta_1, ..., \delta_p \in Y_0$; set $\mathrm{Coder}_0 \mu = (\delta_1^{\mu_1}, \delta_2^{\mu_2}, ..., \delta_q^{\mu_q})$ and $\mathrm{Coder}_0 \lambda = (\delta_1^{\lambda_1}, \delta_2^{\lambda_2}, ..., \delta_p^{\lambda_p})$. Let now $W_{\tilde{\mu}}(L)$ be the subspace of $\mathrm{Super}[X | Y]$ spanned by the set $\{(\boxed{Z} | \mathrm{Coder}_0 \mu);\ Z \in \mathrm{Tab}_\mu(L)\}$, and let $M_\lambda(L)$ be the subspace spanned by the set $\{(T | \mathrm{Coder}_0 \lambda);\ T \in \mathrm{Tab}_\lambda(L)\}$. From [7], it follows that the map

$$(\boxed{Z} | \mathrm{Coder}_0 \mu) \mapsto (\boxed{Z} | \mathrm{Der}_1 \mu), \qquad Z \in \mathrm{Tab}_\mu(L)$$

uniquely defines a $pl(L)$-module isomorphism

$$F\colon W_{\tilde{\mu}}(L) \mapsto S_\mu(L).$$

Furthermore, since

$$\begin{aligned}(\boxed{Z}_{\mu x} | \mathrm{Coder}_0 \lambda) = {} & (\boxed{Z} | \mathrm{Coder}_0 \mu) \\ & \cdot (x^{\lambda_1 - \mu_1} | \delta_1^{\lambda_1 - \mu_1}) \cdot \cdots \cdot (x^{\lambda_p - \mu_p} | \delta_p^{\lambda_p - \mu_p}) \\ & \cdot \binom{\lambda_1}{\mu_1} \cdot \cdots \cdot \binom{\lambda_p}{\mu_p} \cdot (-1)^{f(\lambda, \mu, x)},\end{aligned}$$

$$f(\lambda, \mu, x) = \sum_{h=1}^{p-1} \left(|x| \cdot |\lambda_h - \mu_h| \sum_{k=h+1}^{p} |\mu_k| \right), \qquad Z \in \mathrm{Tab}_\mu(L),$$

it is immediately seen that the map

$$(\boxed{Z} | \mathrm{Coder}_0 \mu) \mapsto (\boxed{Z}_{\mu x} | \mathrm{Coder}_0 \lambda), \qquad Z \in \mathrm{Tab}_\mu(L)$$

uniquely defines a $pl(L)$-module homomorphism

$$G\colon W_{\tilde{\mu}}(L) \mapsto M_{\lambda}(L).$$

Finally, let

$$H\colon M_{\lambda}(L) \mapsto S_{\lambda}(L)$$

be the unique linear operator—with non-trivial kernel—such that $H(T \mid \mathrm{Coder}_0 \lambda) = (T \mid \mathrm{Der}_1 \lambda) = (T \mid \mathrm{Der}(\lambda; \{\beta_i\}))$, $|\beta_i| = 1$, for every $T \in \mathrm{Tab}_{\lambda}(L)$; the operator H can be written, up to a scalar factor, as a product of place polarization operators, and, hence, it is a $pl(L)$-module homomorphism by Proposition 1. Since

$$\mathscr{S}_{\mu x} = k \cdot H \cdot G \cdot F^{-1}, \qquad k \in \mathbb{K},$$

assertion (i) follows.

Let now $|x| = 0$. By applying the Laplace expansions of steps $(\lambda_j - \mu_j)$, $j = 1, 2, ..., p$, the element

$$\mathscr{S}_{\mu x}(\boxed{Z} \mid \mathrm{Der}(\mu; \{\beta_i\}) = (\boxed{Z}_{\mu x} \mid \mathrm{Der}(\lambda; \{\beta_i\})$$

can be rewritten—up to a scalar factor—as

$$(\boxed{Z} \mid \mathrm{Der}_1 \mu) \cdot \prod_{i=1}^{\lambda_1} (x \mid \beta_i)^{\tilde{\lambda}_i - \tilde{\mu}_i}$$

plus a polynomial $\mathbf{P}$ in Super$[X \mid Y]$. Since $\max\{\tilde{\lambda}_i - \tilde{\mu}_i;\ i = 1, 2, ..., \lambda_1\} \leqslant 1$, then $\prod_{i=1}^{\lambda_1} (x \mid \beta_i)^{\tilde{\lambda}_i - \tilde{\mu}_i}$ is non-zero; furthermore, each monomial in $\mathbf{P}$ has multidegree $(d_1, d_2, ..., d_{\lambda})$ different from $(\tilde{\lambda}_1 - \tilde{\mu}_1, \tilde{\lambda}_2 - \tilde{\mu}_2, ..., \tilde{\lambda}_{\lambda_1} - \tilde{\mu}_{\lambda_1})$ in the variables $(x \mid \beta_i)$, $i = 1, 2, ..., \lambda_1$. In consequence of it, the existence of a non-trivial linear relation among the elements

$$(\boxed{Z}_{\mu x} \mid \mathrm{Der}_1 \lambda), \qquad Z \in \mathrm{Tab}_{\mu}(L), \qquad Z \text{ standard}$$

would imply the existence of such a relation among the elements

$$(\boxed{Z} \mid \mathrm{Der}_1 \mu), \qquad Z \in \mathrm{Tab}_{\mu}(L), \qquad Z \text{ standard}$$

contradicting Proposition 2(iv); hence, $\mathscr{S}_{\mu x}$ is a monomorphism.

Assertion (iii) can be proved in an analogous way. ∎

EXAMPLE 2. Let $a, b, c \in L_1$, $d \in L_0$, $x \in X_0$, $\alpha_1, \alpha_2, \alpha_3 \in X_1$, $\beta_1, \beta_2, \beta_3 \in Y_1$, $\mu = (3, 1)$, $\lambda = (5, 3)$. Hence, we have

$$\mathscr{S}_{\mu x}\left(\boxed{\begin{matrix} a & b & d \\ c & & \end{matrix}}\,\middle|\,\begin{matrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_1 & & \end{matrix}\right)$$

$$= \mathscr{S}_{\mu x}\left(D_{a\alpha_1} D_{c\alpha_1} D_{b\alpha_2} D_{c\alpha_3}\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & & \end{pmatrix}\middle|\begin{matrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_1 & & \end{matrix}\right)$$

$$= \mathscr{S}_{\mu x}\left(\left(\begin{matrix} a & b & d \\ c & & \end{matrix}\,\middle|\,\begin{matrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_1 & & \end{matrix}\right) + \left(\begin{matrix} c & b & d \\ a & & \end{matrix}\,\middle|\,\begin{matrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_1 & & \end{matrix}\right)\right)$$

$$= \left(\boxed{\begin{matrix} a & b & d \\ c & & \end{matrix}}_{\mu x}\,\middle|\,\begin{matrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \beta_1 & \beta_2 & \beta_3 & & \end{matrix}\right)$$

$$= \left(\begin{matrix} a & b & d & x & x \\ c & x & x & & \end{matrix}\,\middle|\,\begin{matrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \beta_1 & \beta_2 & \beta_3 & & \end{matrix}\right)$$

$$+ \left(\begin{matrix} c & b & d & x & x \\ a & x & x & & \end{matrix}\,\middle|\,\begin{matrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \beta_1 & \beta_2 & \beta_3 & & \end{matrix}\right)$$

$$= -(a\ b\ d\ x\ x \mid \beta_1\ \beta_2\ \beta_3\ \beta_4\ \beta_5)(c\ x\ x \mid \beta_1\ \beta_2\ \beta_3)$$

$$-(c\ b\ d\ x\ x \mid \beta_1\ \beta_2\ \beta_3\ \beta_4\ \beta_5)(a\ x\ x \mid \beta_1\ \beta_2\ \beta_3).$$

By the Laplace expansion of steps(3, 2), (1, 2) this equals

$$-2(a\ b\ d \mid \beta_1\ \beta_2\ \beta_3)(x \mid \beta_4)(x \mid \beta_5)(c \mid \beta_1)(x \mid \beta_2)(x \mid \beta_3)$$

$$-2(c\ b\ d \mid \beta_1\ \beta_2\ \beta_3)(x \mid \beta_4)(x \mid \beta_5)(a \mid \beta_1)(x \mid \beta_2)(x \mid \beta_3) + \mathbf{P}$$

$$= 2\left(\boxed{\begin{matrix} a & b & d \\ c & & \end{matrix}}\,\middle|\,\begin{matrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_1 & & \end{matrix}\right)(x \mid \beta_4)(x \mid \beta_5)(x \mid \beta_2)(x \mid \beta_3) + \mathbf{P},$$

where $\mathbf{P}$ is a sum of monomials that are not divisible by the monomial $(x \mid \beta_2)(x \mid \beta_3)(x \mid \beta_4)(x \mid \beta_5)$.

In the following, we will denote by $[\mathscr{S}_{\mu x}]$ the image of $S_\mu(L)$ under the homomorphism $\mathscr{S}_{\mu x}$. From Proposition 3 and Theorem 1, it follows:

COROLLARY. (i) *Let* $|x| = 0$, $\max\{\tilde{\lambda}_i - \tilde{\mu}_i;\ i = 1, 2, ..., \lambda_1\} \leqslant 1$; *then* $[\mathscr{S}_{\mu x}]$ *is a simple* $pl(L)$*-submodule of* $S_\lambda(L')$.

(ii) *Let* $|x| = 1$, $\max\{\lambda_i - \mu_i;\ i = 1, 2, ..., p\} \leqslant 1$; *then* $[\mathscr{S}_{\mu x}]$ *is a simple* $pl(L)$*-submodule of* $S_\lambda(L')$.

4. THE BRANCHING THEOREM

Let $n \in \mathbb{N}$. We will denote by $S_\lambda(L'; \operatorname{cont}(x) = n)$ the vector subspace of $S_\lambda(L')$ spanned by the set

$$\{(Z \mid \operatorname{Der}_1 \lambda);\ Z \in \operatorname{Tab}_\lambda(L'),\ \operatorname{cont}(Z; x) = n\}.$$

Furthermore, we will write

$$\mu \subset_{0n} \lambda$$

whenever $\mu \in H(L)$, $\mu \subseteq \lambda$, $|\lambda| - |\mu| = n$, and $\max\{\tilde{\lambda}_i - \tilde{\mu}_i;\ i = 1, 2, ..., \lambda_1\} \leqslant 1$; similarly, we will write

$$\mu \subset_{1n} \lambda$$

whenever $\mu \in H(L)$, $\mu \subseteq \lambda$, $|\lambda| - |\mu| = n$, and $\max\{\lambda_i - \mu_i; i = 1, 2, ..., p\} \leqslant 1$.
We have:

PROPOSITION 5. (i) *$S_\lambda(L'; \mathrm{cont}(x) = n)$ is a $pl(L)$-submodule of $S_\lambda(L')$.*

(ii) *Let $|x| = 0$. Then $S_\lambda(L'; \mathrm{cont}(x) = n) \neq (0)$ if and only if $\{\mu; \mu \subset_{0n} \lambda\} \neq \varnothing$.*

(iii) *Let $|x| = 1$. Then $S_\lambda(L'; \mathrm{cont}(x) = n) \neq (0)$ if and only if $\{\mu; \mu \subset_{1n} \lambda\} \neq \varnothing$.*

THEOREM 2. *$S_\lambda(L'; \mathrm{cont}(x) = n)$ is a semisimple $pl(L)$-module. The following complete decompositions hold:*

(i) *Let $|x| = 0$. Then $S_\lambda(L'; \mathrm{cont}(x) = n) = \bigoplus_{\mu \subset_{0n} \lambda} [\mathscr{S}_{\mu x}]$.*

(ii) *Let $|x| = 1$. Then $S_\lambda(L'; \mathrm{cont}(x) = n) = \bigoplus_{\mu \subset_{1n} \lambda} [\mathscr{S}_{\mu x}]$.*

COROLLARY. (i) *Let $|x| = 0$. The set*

$$\{(\boxed{Z}_{\mu x} \mid \mathrm{Der}_1 \lambda);\ \mu \subset_{0n} \lambda,\ Z \in \mathrm{Tab}_\mu(L),\ Z \textit{ standard}\}$$

is a basis of $S_\lambda(L'; \mathrm{cont}(x) = n)$.

(ii) *Let $|x| = 1$. The set*

$$\{(\boxed{Z}_{\mu x} \mid \mathrm{Der}_1 \lambda);\ \mu \subset_{1n} \lambda,\ Z \in \mathrm{Tab}_\mu(L),\ Z \textit{ standard}\}$$

is a basis of $S_\lambda(L'; \mathrm{cont}(x) = n)$.

The preceding results can be proved in the following way. Let $\leqslant$ be a linear order on $L' = L \cup \{x\}$ such that x is the *greatest* element. Proposition 5 follows from Proposition 2 by considering diagrams in $\mathrm{Tab}_\lambda(L')$ that are standard with respect to the linear order $\leqslant$. Furthermore, the simple $pl(L)$-modules $[\mathscr{S}_{\mu x}]$ that appear in (i) of Theorem 2 are mutually non-isomorphic [4, Corollary 4]; thus, their sum is a direct sum. On the other hand, the sets

$$\{Z \in \mathrm{Tab}_\mu(L);\ \mu \subset_{0n} \lambda,\ Z \leqslant\text{-standard}\}$$

and

$$\{S \in \mathrm{Tab}_\lambda(L');\ \mathrm{cont}(S; x) = n,\ S \leqslant\text{-standard}\}$$

are equicardinal and, hence, assertion (i) follows again from Proposition 2. Assertion (ii) of Theorem 2 can be proved in an analogous way.

Since $S_\lambda(L') = \bigoplus_{n\in\mathbb{N}} S_\lambda(L'; \mathrm{cont}(x)=n)$ as a $pl(L)$-module, one immediatly derives the following version of the Branching Rules.

COROLLARY. *$S_\lambda(L')$ is a semisimple $pl(L)$-module. The following complete decompositions hold:*

(i) *Let $|x|=0$. Then*

$$S_\lambda(L') = \bigoplus_\mu [\mathscr{S}_{\mu x}],$$

where the summation ranges over the set of all the shapes $\mu=(\mu_1, ..., \mu_q)$, such that $\mu\in H(L)$, and $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \cdots$.

(ii) *Let $|x|=1$. Then*

$$S_\lambda(L') = \bigoplus_\mu [\mathscr{S}_{\mu x}],$$

where the summation ranges over the set of all the shapes $\mu=(\mu_1, ..., \mu_q)$, such that $\mu\in H(L)$, and $\tilde{\lambda}_1 \geqslant \tilde{\mu}_1 \geqslant \tilde{\lambda}_2 \geqslant \tilde{\mu}_2 \geqslant \cdots$.

5. PIERI'S FORMULA

Let $\mathscr{L}$ be a Lie superalgebra and let M and N be $\mathscr{L}$-modules; the tensor product $M\otimes N$ turns out to be an $\mathscr{L}$-module with respect to the action defined as

$$\rho(x\otimes y) = \rho\cdot x\otimes y + (-1)^{|\rho|\,|x|}\, x\otimes \rho\cdot y,$$

for every $x\in M$, $y\in N$, $\rho\in\mathscr{L}$, x, ρ homogeneous. The tensor product $M\otimes N$ is isomorphic, as an $\mathscr{L}$-module, to $N\otimes M$ under the map

$$x\otimes y \mapsto (-1)^{|x|\,|y|}\, y\otimes x,$$

$x\in M$, $y\in N$, $\rho\in\mathscr{L}$, x, y homogeneous.

Our next aim is to derive a version of Pieri's formula which holds for Schur $pl(L)$-modules, that is, to describe a complete decomposition of the tensor product $S_\lambda(L)\otimes S_\theta(L)$ in the case when λ is any Young shape in $H(L)$ and θ is a "row shape" or a "column shape."

Let $n\in\mathbb{Z}^+$; we write $\mathbf{n}$ for the row shape $\mathbf{n}=(n)$. Given any shape $\lambda=(\lambda_1, ..., \lambda_p)$, we set $\mathbf{n}\otimes\lambda = (n+\lambda_1, \lambda_1, ..., \lambda_p)$.

Let now $\omega \in \mathrm{Mon}(L)$ be a word of length n, $Z = (\omega_1, ..., \omega_p)$ a diagram in $\mathrm{Tab}_\lambda(L)$ and x an element in $X - L$ such that $|x| = 0$; we will denote by $\omega \otimes Z$ the diagram $(x^{\lambda_1}\omega, \omega_1, ..., \omega_p)$. Clearly $\omega \otimes Z \in \mathrm{Tab}_{\mathbf{n} \otimes \lambda}(L')$, $L' = L \cup \{x\}$.

PROPOSITION 6. *The map*

$$\begin{aligned}(\omega \mid \mathrm{Der}_1 \mathbf{n}) &\otimes (Z \mid \mathrm{Der}_1 \lambda) \\ &\mapsto (-1)^{|n| \cdot (|\omega_1| + \cdots + |\omega_p|)} (\omega \otimes Z \mid \mathrm{Der}_1 \mathbf{n} \otimes \lambda)\end{aligned}$$

with $\omega \in \mathrm{Mon}(L)$, $\mathrm{length}(\omega) = n$, $Z \in \mathrm{Tab}_\lambda(L)$, *uniquely defines a* $pl(L)$-*module isomorphism*

$$\Phi: S_{\mathbf{n}}(L) \otimes S_\lambda(L) \to S_{\mathbf{n} \otimes \lambda}(L'; \mathrm{cont}(x) = \lambda_1).$$

Proof. Let $\leqslant'$ be a linear order on $L' = L \cup \{x\}$ such that x is the *smallest* element. The map Φ induces a bijection between the sets

$$\begin{aligned}&\{(S' \mid \mathrm{Der}_1 \mathbf{n}) \otimes (S \mid \mathrm{Der}_1 \lambda);\ S' \in \mathrm{Tab}_{\mathbf{n}}(L),\ S \in \mathrm{Tab}_\lambda(L), \\ &S \text{ and } S' \leqslant'\text{-standard}\}\end{aligned} \tag{1}$$

and

$$\begin{aligned}&\{(-1)^{|n| \cdot (|\omega_1| + \cdots + |\omega_p|)} (S' \otimes S \mid \mathrm{Der}_1 \mathbf{n} \otimes \lambda); \\ &S' \in \mathrm{Tab}_{\mathbf{n}}(L),\ S \in \mathrm{Tab}_\lambda(L),\ S \text{ and } S' \leqslant'\text{-standard}\},\end{aligned} \tag{2}$$

that are bases of $S_{\mathbf{n}}(L) \otimes S_\lambda(L)$ and $S_{\mathbf{n} \otimes \lambda}(L'; \mathrm{cont}(x) = \lambda_1)$, respectively. Hence, Φ uniquely defines a $\mathbb{K}$-linear isomorphism. Furthermore, given two *arbitrary* diagrams $C \in \mathrm{Tab}_{\mathbf{n}}(L)$, $Z \in \mathrm{Tab}_\lambda(L)$, the element

$$(C \mid \mathrm{Der}_1 \mathbf{n}) \otimes (Z \mid \mathrm{Der}_1 \lambda) \tag{3}$$

is uniquely expanded as a linear combination of the basis elements (1), by applying the straightening relations of $S_{\mathbf{n}}(L)$ and $S_\lambda(L)$ with respect to the linear order induced by $\leqslant'$; by applying the straightening relations of $S_{\mathbf{n} \otimes \lambda}(L'; \mathrm{cont}(x) = \lambda_1)$ with respect to the linear order induced by $\leqslant'$, the image of (3) under the map Φ is uniquely expanded as a linear combination of the basis elements (2), with the *same* coefficients. Hence

$$\begin{aligned}\Phi((C \mid \mathrm{Der}_1 \mathbf{n}) &\otimes (Z \mid \mathrm{Der}_1(\lambda)) \\ &= (-1)^{|n| \cdot (|\omega_1| + \cdots + |\omega_p|)} (C \otimes Z \mid \mathrm{Der}_1 \mathbf{n} \otimes \lambda)\end{aligned}$$

for every $C \in \mathrm{Tab}_{\mathbf{n}}(L)$, $Z \in \mathrm{Tab}_\lambda(L)$.

The fact that Φ is a $pl(L)$-module isomorphism can be checked by straightforward computations. ▌

Let now $\tilde{\mathbf{n}}$ be the conjugate shape of $\mathbf{n}$. We denote by $\lambda \otimes \tilde{\mathbf{n}}$ the conjugate shape of $\mathbf{n} \otimes \tilde{\lambda}$. Given a diagram $Z = (\omega_1, ..., \omega_p) \in \mathrm{Tab}_\lambda(L)$, a diagram $C = (x_1, ..., x_n)$, $x_i \in L$, and an element $x \in X - L$, $|x| = 1$, we will denote by $Z \otimes C$ the diagram $(x\omega_1, ..., x\omega_p, x_1, ..., x_n)$. Clearly $Z \otimes C \in \mathrm{Tab}_{\lambda \otimes \tilde{\mathbf{n}}}(L')$, $L' = L \cup \{x\}$.

PROPOSITION 7. *The map*

$$\begin{aligned}(Z \mid \mathrm{Der}_1 \lambda) &\otimes (C \mid \mathrm{Der}_1 \tilde{\mathbf{n}}) \\ &\mapsto (-1)^{g(\lambda, \mu, Z)} (Z \otimes C \mid \mathrm{Der}_1 \lambda \otimes \tilde{\mathbf{n}})\end{aligned}$$

with $Z \in \mathrm{Tab}_\lambda(L)$, $C \in \mathrm{Tab}_{\tilde{\mathbf{n}}}(L)$, *and*

$$g(\lambda, \mu, Z) = \sum_{h=1}^{p} |\omega_h| \cdot |h| + |\lambda_1 + \cdots + \lambda_p + p| \cdot (|x_1| + \cdots + |x_n|)$$

uniquely defines a $pl(L)$*-module isomorphism*

$$\Psi\colon S_\lambda(L) \otimes S_{\tilde{\mathbf{n}}}(L) \to S_{\lambda \otimes \tilde{\mathbf{n}}}(L'; \mathrm{cont}(x) = p).$$

The proof of Proposition 7 is essentially the same as that of Proposition 6. However, the fact that the map Ψ "preserves the straightening relations" is a little more delicate to prove. Specifically, let $(\omega_1, ..., \omega_p) \in \mathrm{Tab}_\lambda(L)$; the generic straightening relation of $S_\lambda(L)$ can be written as

$$\sum_{\omega_{(1)}\omega_{(2)}} \left(\begin{array}{c|c} \omega_1 & \\ \vdots & \\ \omega_{h-1} & \\ v_1 \omega_{(1)} & \\ \omega_{(2)} v_2 & \mathrm{Der}_1 \lambda \\ \omega_{h+2} & \\ \vdots & \\ \omega_p & \end{array} \right) = 0, \qquad h = 1, ..., p-1,$$

where $\omega_h = v_1 \omega'$, $\omega_{h+1} = \omega'' v_2$, $v_1, v_2 \in \mathrm{Mon}(L)$ and

$$\sum \omega_{(1)} \otimes \omega_{(2)} = \Delta(\omega)$$

is the image of the element $\omega = \omega' \omega''$, $\mathrm{length}(\omega) > \lambda_h$, under the diagonalization map of the Hopf algebra $\mathrm{Super}[L]$ [11, 13].

Set $\lambda' = (\lambda_1 + 1, ..., \lambda_p + 1)$, We have

$$\sum_{\omega_{(1)}\omega_{(2)}} (-1)^{|\omega_1| + \cdots + |\omega_{h-1}|\, |h-1| + (|v_1| + |\omega_{(1)}|)\, |h|} \times (-1)^{(|\omega_{(2)}| + |v_2|)\, |h+1| + |\omega_{h+2}|\, |h+2| + \cdots + |\omega_p|\, |p|} \times \left(\begin{array}{c|c} x\omega_1 & \\ \vdots & \\ x\omega_{h-1} & \\ xv_1\omega_{(1)} & \mathrm{Der}_1 \lambda' \\ x\omega_{(2)}v_2 & \\ x\omega_{h+2} & \\ \vdots & \\ x\omega_p & \end{array} \right)$$

$$= (-1)^{\sum_{h=1}^{p} |\omega_h|\, |h|} \sum_{\bar{\omega}_{(1)}\bar{\omega}_{(2)}} \left(\begin{array}{c|c} x\omega_1 & \\ \vdots & \\ x\omega_{h-1} & \\ xv_1\bar{\omega}_{(1)} & \mathrm{Der}_1 \lambda' \\ \bar{\omega}_{(2)}v_2 & \\ x\omega_{h+2} & \\ \vdots & \\ x\omega_p & \end{array} \right),$$

where $\bar{\omega} = \omega' x \omega''$ and

$$\Delta(\bar{\omega}) = \sum \bar{\omega}_{(1)} \otimes \bar{\omega}_{(2)}$$

in the Hopf algebra Super$[L']$; the last sum equals 0, since it is a straightening relation in $S_{\lambda'}(L')$.

COROLLARY (Pieri's Formula). (i) *$S_{\mathbf{n}}(L) \otimes S_\lambda(L)$ is a semisimple $pl(L)$-module. The following isomorphism holds*:

$$S_{\mathbf{n}}(L) \otimes S_\lambda(L) \cong \bigoplus_{\mu} S_\mu(L),$$

where the summation ranges over the set

$$\{\mu \in H(L);\, |\mu| = |\lambda| + n,\, \tilde{\lambda}_i \leqslant \tilde{\mu}_i \leqslant \tilde{\lambda}_i + 1\}.$$

(ii) *$S_{\lambda}(L) \otimes S_{\tilde{n}}(L)$ is a semisimple $pl(L)$-module. The following isomorphism holds*:

$$S_{\lambda}(L) \otimes S_{\tilde{n}}(L) \cong \bigoplus_{\mu} S_{\mu}(L),$$

where the summation ranges over the set

$$\{\mu \in H(L);\ |\mu| = |\lambda| + n,\ \lambda_i \leqslant \mu_i \leqslant \lambda_i + 1\}.$$

6. A Remark on the Schur–Weyl Duality (The Involution of the Ring of Symmetric Functions)

Let $L \subset X$ be a proper $\mathbb{Z}_2$-graded set. A *dual* set of L is a subset $L^* \subset X$ together with a bijection $* : L \to L^*$ such that, writing x^* for the image of $x \in L$ under the map $*$, the following condition holds:

$$|x^*| = |x| + 1, \qquad \text{for every} \quad x \in L.$$

The bijection $* : L \to L^*$ uniquely defines an isomorphism from $\mathrm{Mon}(L)$ to $\mathrm{Mon}(L^*)$; the image in $\mathrm{Mon}(L^*)$ of a word $\omega \in \mathrm{Mon}(L)$ will be consistently denoted by ω^*. Given a diagram $T = (\omega_1, ..., \omega_p)$, $\omega_i \in \mathrm{Mon}(L)$, we set $T^* = (\omega_1^*, ..., \omega_p^*)$.

The Lie superalgebra $pl(L^*)$ is canonically isomorphic to $pl(L)$ and, hence, a $pl(L^*)$-module can be regarded as a $pl(L)$-module and vice versa, in the obvious way.

Let now $\lambda \in H(L)$. The Schur module $S_{\tilde{\lambda}}(L^*)$ of shape $\tilde{\lambda}$ over the set L^* is isomorphic, as a $pl(L)$-module, to $S_{\lambda}(L)$. The $\mathbb{K}$-vector space $\mathrm{Hom}_{pl(L)}(S_{\lambda}(L), S_{\tilde{\lambda}}(L^*))$ is 1-dimensional; its elements are the scalar multiples of the isomorphism

$$\theta\colon S_{\lambda}(L) \to S_{\tilde{\lambda}}(L^*)$$

defined by extending the map

$$(T \mid \mathrm{Der}_1 \lambda) \to (\boxed{\tilde{T}^*} \mid \mathrm{Der}_1 \tilde{\lambda}),$$

for every $T \in \mathrm{Tab}_{\lambda}(L)$ [7, 8].

Moreover, we have:

Proposition 8. *The restricted map*

$$(T \mid \mathrm{Der}_1 \lambda) \to (\boxed{\tilde{T}^*} \mid \mathrm{Der}_1 \tilde{\lambda}),$$

with $T \in \mathrm{Tab}_\lambda(L)$, *cont*$(T; x) = n$, *uniquely defines a* $pl(L - \{x\})$-*module isomorphism*

$$\theta\colon S_\lambda(L; \mathrm{cont}(x) = n) \to S_{\tilde{\lambda}}(L^*; \mathrm{cont}(x^*) = n).$$

Using Proposition 8, assertion (ii) of Theorem 2 can be derived—up to isomorphism—from assertion (i) and vice versa.

Specifically, let L be a proper set, $x \notin L$, $|x| = 1$, $L' = L \cup \{x\}$. Set $L'^* = L^* \cup \{x^*\}$. By Proposition 8, we have

$$S_\lambda(L'; \mathrm{cont}(x) = n) = \theta[S_{\tilde{\lambda}}(L'^*; \mathrm{cont}(x^*) = n)].$$

Applying Theorem 2(i), we have the decomposition

$$\theta[S_{\tilde{\lambda}}(L'^*; \mathrm{cont}(x^*) = n)] = \bigoplus_{\tilde{\mu} \subset_{0n} \tilde{\lambda}} \theta[[\mathscr{S}_{\tilde{\mu}x^*}]].$$

Since $\theta[[\mathscr{S}_{\tilde{\mu}x^*}]] \cong [\mathscr{S}_{\tilde{\mu}x^*}] \cong S_{\tilde{\mu}}(L^*) \cong S_\mu(L)$, we have

$$S_\lambda(L'; \mathrm{cont}(x) = n) \cong \bigoplus_{\mu \subset_{1n} \lambda} S_\mu(L).$$

Finally, it should be noted that Pieri's Formulae reduce to a single formula in the superalgebraic setting; as a matter of fact, one can deduce formula (ii) from formula (i) by using the isomorphism θ, and vice versa. Furthermore, the classical formulae for the general linear group $GL(V)$ are seen to be special cases of our formula (i), by setting $L = L_1$ and $L = L_0$, respectively. Indeed, if $L = L_1$—that is, $V = V_1$—then $S_n(L) \cong \Lambda^n V$, $S_\lambda(L) \cong L_\lambda V$, and, if $L = L_0$, then $S_n(L) \cong \mathrm{Sym}_n V$, $S_\lambda(L) \cong L_{\tilde{\lambda}} V$, where $L_\lambda V$ denotes the Schur $GL(V)$-module of shape λ according to the notation of [1].

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